MATH 2060 Tutorial 12

Section 9.3

Q1 Test the following series for convergence and for absolutely convergence

(c) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} n}{n+2}$ (d) $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{\ln n}{n}$

Sol (c) Let $a_n = \frac{(-1)^{n+1} n}{n+2}$

Then $\lim_{n\to\infty} a_{2n} = -1$ and $\lim_{n\to\infty} a_{2n+1} = 1$

Therefore, $\lim_{n\to\infty} a_n$ does not converge to 0, which implies the series Σa_n is divergent by n-th term test

(d) Let $a_n = (-1)^{n+1} \frac{\ln n}{n}$ and $z_n = \frac{\ln n}{n}$

Write $f(x) = \frac{\ln x}{x}$ and note that

$$f'(x) = \frac{1-\ln x}{x^2} < 0, \quad \forall x \geq 4$$

It follows that z_n is decreasing for $n \geq 4$ and $\lim_{n\to\infty} z_n = 0$

By the Alternating Series Test, Σa_n is convergent

However, $|a_n| = \frac{\ln n}{n} > \frac{1}{n}$, $\forall n \geq 1$

Since $\Sigma \frac{1}{n}$ is divergent and by the Comparison Test, $\Sigma |a_n|$ is divergent

Therefore Σa_n is not absolutely convergent

Section 9.4

Q5 Show that the radius of convergence R of the power series $\sum a_n x^n$ is given by $\lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right|$, whenever this limit exists

Sol: Let $L = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right|$

The existence of the limit is allowed for $L \in [0, +\infty]$

Case ①: When $L \in (0, +\infty)$, for $|x| < \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| = L$

$$\lim_{n\to\infty} \frac{|a_{n+1} x^{n+1}|}{|a_n x^n|} = |x| \cdot \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < L \cdot \frac{1}{L} = 1$$

Similarly for $|x| > L$, we have $\lim_{n\to\infty} \frac{|a_{n+1} x^{n+1}|}{|a_n x^n|} > 1$

By the Ratio Test, $\sum a_n x^n$ is convergent for $|x| < L$ and is divergent for $|x| > L$

Cauchy-Hadamard Theorem implies $L = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right|$ is the radius of convergence

Case ②: When $L = 0$, $\forall\, |x| > 0$

Then $\left|\frac{a_n x^n}{a_{n+1} x^{n+1}}\right| = \left|\frac{a_n}{a_{n+1}}\right| \cdot \frac{1}{|x|}$

Since $\lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| = 0$, we can find a large number $N > 0$ such that $\left|\frac{a_n}{a_{n+1}}\right| < |x|$ for all $n > N$

Therefore, $\left|\frac{a_n x^n}{a_{n+1} x^{n+1}}\right| < 1$ for all $n > N$

It implies $|a_n x^n|$ is increasing, and $\lim_{n\to\infty} a_n x^n$ does not converge to 0, thus, $\sum a_n x^n$ is divergent for any $|x| > 0$

It follows that the radius of convergence $R = L = 0$

Case ③ When $L = +\infty$, for any real number x s.t. $|x| > 0$

$$\lim_{n\to\infty} \left| \frac{a_{n+1} x^{n+1}}{a_n x^n} \right| = |x| \cdot \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = |x| \cdot 0 = 0$$

By the Ratio Test, $\sum a_n x^n$ is convergent

Moreover, for the case $x = 0$, $\sum a_n x^n \equiv 0$ is convergent

It follows that $\sum a_n x^n$ is convergent for any $x \in \mathbb{R}$, then the radius of convergence $R = L = +\infty$

Section 9.4

Q6 Determine the radius of convergence of the series

$\sum a_n x^n$, where a_n is given by

(a) $\frac{1}{n^n}$ (c) $\frac{n^n}{n!}$

Sol: (a) $\limsup_{n\to\infty} \left(\left|\frac{1}{n^n}\right|^{\frac{1}{n}} \right) = \limsup_{n\to\infty} \frac{1}{n} = 0$

Then $R = +\infty$

(c) $\lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty} \left| \frac{n^n}{n!} \cdot \frac{(n+1)!}{(n+1)^{n+1}} \right|$

$= \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n$

$= e^{-1}$

It follows from Q5 that $R = e^{-1}$

Section 9.4

11. Prove that if f is defined for $|x| < r$ and if there exists a constant B such that $|f^{(n)}(x)| \le B$ for all $|x| < r$ and $n \in \mathbb{N}$, then the Taylor series expansion

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

converges to $f(x)$ for $|x| < r$.

Sol: By Taylor's Theorem (Thm 6.4.1), $\forall N \in \mathbb{N}$,

$\forall |x| < r$, there exists c between 0 and x, $|c| < |x|$, s.t.

$$\left| f(x) - \sum_{n=0}^{N} \frac{f^{(n)}(0)}{n!} x^n \right| = \left| \frac{f^{(N+1)}(c)}{(N+1)!} x^{N+1} \right| \le \frac{B}{(N+1)!} |x|^{N+1} < B \frac{r^{N+1}}{(N+1)!}$$

for some constant B

Take $a_n = \frac{r^n}{n!} > 0$, note that

$$\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \lim_{n \to \infty} \frac{r}{n+1} = 0,$$ which implies a_n is decreasing to 0

Hence, $$\lim_{N \to \infty} \left| f(x) - \sum_{n=0}^{N} \frac{f^{(n)}(0)}{n!} x^n \right| = 0,$$

which exactly gives the convergence of the Taylor expansion